THE

ELEMENTS

OF THE

CONIC SECTIONS

WITH

The Sections

OF THE

CONOIDS.

By Hustler

SECOND EDITION.

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THE

ELEMENTS

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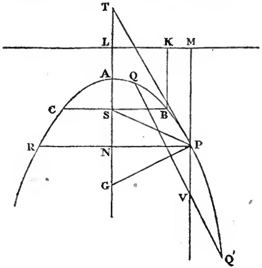
CONIC SECTIONS.

On the Parabola.

DEFINITIONS.

1. If a straight line SP revolve about a fixed point S, and a point P be taken in it, so that SP may always be equal to the perpendicular distance PM of P



from another straight line LM given in position, the curve RAP, passing through all the points P, is called a *Parabola*.

2. The point S is called the *Focus*; LM the *Directrix*; a straight line LSN perpendicular to the directrix through the focus, the *Axis*; and the point A, the *Vertex*.

3. Any line PNR perpendicular to the axis and terminated both ways by the curve, is called an *Ordinate* to the axis; the part AN of the axis between the vertex and ordinate, the *Abscissa*; and the ordinate BC through the focus, the *Latus Rectum*. Also it is manifest from the construction, that the parts of the curve on each side of the axis are similar and equal, and that every ordinate PR is bisected by the axis.

4. Any line MPV parallel to the axis, is called a *Diameter;* a line QVQ' parallel to the tangent at any point P, is called an *Ordinate* to the point P or diameter PV and the part PV of the diameter, the *Abscissa*. Also that ordinate, which passes through the focus, is called the *Parameter*.

5. A straight line PG perpendicular to the tangent at any point P and terminated by the axis, is called a *Normal*; and the part NG of the axis, the *Subnormal*.

6. The part of any diameter between one of it's own ordinates and the intersection of a tangent at the extremity of the ordinate, is called the *Subtangent*. Thus NT is the subtangent of the axis.

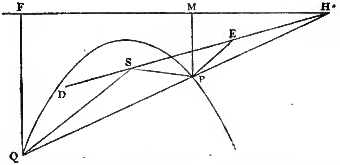
Prop. I.

The latus rectum BC *is equal to* 4 AS *.

Draw BK perpendicular to the directrix;
Then $SA = AL$, and $SB = BK$ (Def. 1.) $= SL = 2AS$;
So $SC = 2AS$, and therefore $BC = 4AS$.

Prop. II.

If a straight line QP *cut the parabola in* P, Q, *and the directrix in* H, *then* HSD *drawn through the focus makes equal angles with the focal distances* SP, SQ.



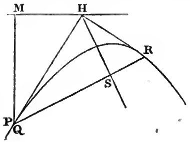
Draw PM, QF, perpendicular to the directrix, and PE parallel to SQ; then the triangles HPE, HQS, and the triangles HPM, HQF, are similar.

Hence $PE : QS :: HP : HQ :: PM : QF$;
But $QS = QF$, and $\therefore PE = PM = SP$.
Hence $\angle PSE = \angle PES = \angle QSD$. (Eucl. 29. I.)

* The reference is always to the Figure immediately preceding.

COR. 1. If any straight line HP, not parallel to the axis, cuts a parabola in one point P, it will cut it again. For with center P and radius PS or PM, which is necessarily less than PH, suppose a circle described cutting HS in E. In HP, produced if necessary, take HQ a fourth proportional to HE, HS, HP; and draw PM, QF, perpendicular to the directrix. Then Q is a point in the curve, for $SQ=QF$, by similar triangles, as in the proposition.

COR. 2. If PQ move parallel to itself until the



points P and Q coincide, and HQ touches the parabola, the $\angle PSQ$ vanishes, and each of the angles PSE, QSD is a right angle. Conversely, if SH be drawn perpendicular to any focal distance SP, cutting the directrix in H, then HP being joined touches the curve at P.

COR. 3. Let PS meet the curve again in R; then HR being joined touches at R. The tangents therefore at the extremities of any parameter meet in the directrix. In the case of the latus rectum, the tangents meet in the intersection of the axis with the directrix, and are at right angles to each other.

Prop. III.

A tangent PH *bisects the angle* SPM.

For $\angle HSP$ is a right angle (Prop. 2. Cor. 2.); and HP is common to the two right-angled triangles HSP, HMP; also $SP = PM$; therefore the triangles are equal, and $\angle SPH = \angle MPH$.

Cor. A tangent at the vertex is perpendicular to the axis, and parallel to all the ordinates of the axis.

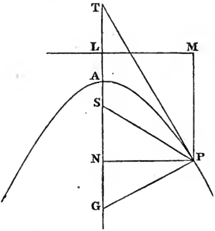
Prop. IV.

The focal distance SP *is equal to*

i. *The abscissa of the axis* AN+AS.

ii. *The distance* ST *of the tangent's intersection with the axis.*

iii. *The distance* SG *of the normal's intersection with the axis.*



For i. $SP = PM = NL = AN + AL = AN + AS.$

II. $\angle SPT = \angle MPT =$ altern. $\angle STP$; $\therefore SP = ST$.

III. The right angle $GPT = \angle PGT + \angle GTP$ $= \angle PGS + \angle SPT$; take away $\angle SPT$, and $\angle SPG = \angle SGP$; $\therefore SP = SG$.

PROP. V.

The subnormal NG = 2AS.

For $SG = SP = AN = AS$ (Prop. 4.) $= SN + 2AS$;
Take away SN, and $NG = 2AS$.

PROP. VI.

The subtangent NT *of the axis is double of the abscissa* AN.

For $ST = SP = AN + AS$ (Prop. 4.);
Take away AS, and $AT = AN$.

PROP. VII.

The rectangle by the latus rectum and the abscissa AN *of the axis is equal to the square of the semi-ordinate* NP.

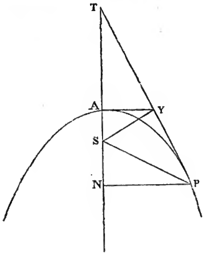
For $4AS \times AN + SN^2 = NL^2$ (Eucl. 8. II.)
$= SP^2 = SN^2 + NP^2$;
Take away SN^2, and $4AS \times AN = NP^2$.

COR. 1. $AN \propto NP^2$.

COR. 2. $4AS\ (L) : NP :: NP : AN$;
$\therefore L : 2NP\ (PR) :: NP : 2AN\ (NT.)$

Prop. VIII.

A perpendicular SY *upon any tangent* PT *intersects* PT *in the tangent at the vertex* A.



Draw the tangent AY, which is parallel to PN (Prop. 3. Cor.), and let PT intersect AY in Y: join SY.

Then $AN=AT$ (Prop. 6.), and $\therefore PY=TY$. (Eucl. 2. VI.) Also $SP=ST$, and SY is common: Hence $\angle SYP = \angle SYT$, and SY is perpendicular to PT.

Cor. The triangles SAY, SPY being similar,

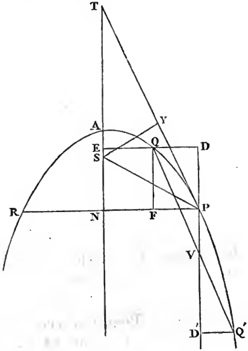
$$SP : SY :: SY : SA,$$

$$\text{and } SP : SA :: SP^2 : SY^2.$$

Also $SY^2 = SP \times SA$, and $\propto SP$.

Prop. IX.

The rectangle by the latus rectum (L) *and the part* QF *of any diameter cut off by an ordinate to the axis, is equal to the rectangle by the segments* PF, FR *of the ordinate.*



Draw QE perpendicular to the axis;
Then $PF \times FR = NP^2 - NF^2$ (Eucl. 5. II.)
$= NP^2 - QE^2 = L \times AN - L \times AE$ (Prop. 7.)
$= L \times EN = L \times QF.$

Prop. X.

If from either extremity of an ordinate QQ′ *to any diameter a perpendicular* QD *be let fall upon the diameter, the square of* QD *is equal to the rectangle by the latus rectum and the abscissa* PV.

The △s QVD, PNT, SPY are manifestly similar;
Hence $QD : DV :: NP : NT$,
$:: L : PR$ (Prop. 7. Cor. 2.)
$\therefore L \times DV = PR \times QD = PR \times PF$
Also $L \times DP = L \times QF = RF \times PF$ (Prop. 9.)
Hence $L \times PV = PF^2$ (Eucl. 3. II.) $= QD^2$.
In the same manner if $Q'D'$ be the perpendicular from the other extremity Q', it may be shewn that $L \times PV = Q'D'^2$.

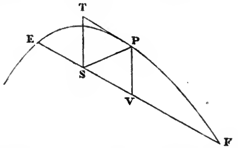
Cor. $QD = Q'D'$, and the similar triangles QVD, $Q'VD'$ are therefore equal; hence $QV = Q'V$; *i. e.* a diameter bisects all it's own ordinates.

Prop. XI.

The rectangle by four times the focal distance SP, *and the abscissa* PV *is equal to the square of the semi-ordinate* QV.

For $QV^2 : QD^2 :: SP^2 : SY^2$
Or $QV^2 : L \times PV :: SP : SA$ (Prop. 8. Cor.)
$:: 4SP : L$;
$\therefore 4SP \times PV = QV^2$.

Cor. 1. Let EF be the parameter. Then $SP = ST$ (Prop. 4.) $= PV$; also $EV^2 = 4SP \times PV = 4SP^2$.

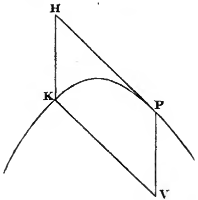


Hence $EV = 2SP$, and the parameter $= 4SP$.

Cor. 2. The rectangle by the parameter and abscissa of any diameter = the square of the corresponding semi-ordinate.

Cor. 3. The abscissa varies as the square of the semi-ordinate.

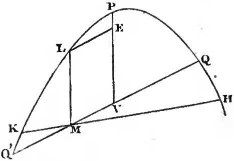
Cor. 4. If HK be drawn parallel to the axis



from any point H in a tangent, then $HK \propto PH^2$. For, if the parallelogram $PHKV$ be completed, $HK =$ the abscissa PV, and $PH =$ the semi-ordinate KV.

Prop. XII.

The rectangle by the parameter (P) *of any diameter and the part* LM *of any other diameter cut off by an ordinate* QQ′ *of the first, is equal to the rectangle by the segments* QM, MQ′ *of the ordinate.*



Draw LE parallel to QQ'.

Then $QM \times MQ' = QV^2 - VM^2$ (Eucl. 5. II.)
$= QV^2 - LE^2 = P \times PV - P \times PE$
(Prop. 11. Cor. 2.) $= P \times EV = P \times LM$.

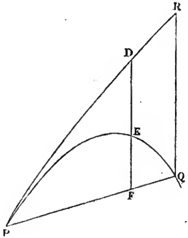
Cor. 1. Let any other ordinate HK, whose parameter is P', intersect QQ' in M: then

$$QM \times MQ' : HM \times MK :: P \times LM : P' \times LM$$
$$:: P : P'.$$

Cor. 2. If the lines QQ', HK, move parallel to themselves and intersect each other, either within or without the parabola, the rectangles of their segments are always in the same constant ratio.

Prop. XIII.

If an ordinate PQ *and a tangent* PR *be drawn from the same point* P, *any diameter* DEF *terminated by the ordinate and tangent is divided by the curve in the same proportion in which itself divides the ordinate.*



Draw QR parallel to the axis;

Then $DE : QR :: PD^2 : PR^2$ (Prop. 11. Cor. 4.)

$:: PF^2 : PQ^2$,

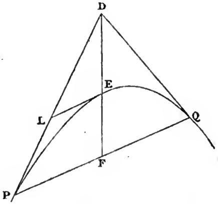
And $QR : DF :: PQ : PF$;

$\therefore DE : DF :: PF : PQ$,

Div^{do}. $DE : EF :: PF : FQ$.

Cor. 1. If DEF bisects PQ, it is then the diameter of which PQ is an ordinate, and DEF is the

subtangent (Def. 6.): but then $DE = EF$, *i. e.* the



subtangent is double of the abscissa in all cases.

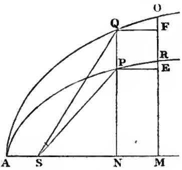
Cor. 2. If a tangent be drawn at Q, it must meet the diameter DEF in the same point D.

Cor. 3. The tangent at E is parallel to the ordinate PQ; hence $PL = LD$.

Prop. XIV.

If two parabolas, PAN, QAN, *whose latera recta are* L, L′, *respectively, have a common axis and vertex, the areas* PAN, QAN, *cut off by a common ordinate* QPN, *are in the subduplicate ratio of the latera recta.*

Draw OM parallel to QN and PE, QF perpendicular



to OM. Then (Eucl. 1. vi.)

$$\begin{aligned} \text{rectangle } PM : \text{rectangle } QM &:: PN : QN \\ &:: \sqrt{L \times AN} : \sqrt{L' \times AN} \\ &:: \sqrt{L} \quad : \sqrt{L'}; \end{aligned}$$

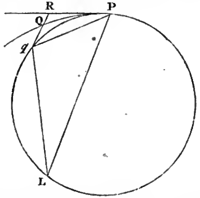
which constant ratio holds for all the corresponding rectangles thus inscribed in the areas PAN, QAN, and *componendo* for the sums of them in each, into how many parts soever the abscissa AM is divided. Let the number of the parts MN be increased, and the magnitude of each diminished, indefinitely. Then in the limit the sums of the inscribed rectangles are respectively equal to the areas PAN, QAN. (Newton. Lem. 4.) The areas therefore are in the constant proportion of $\sqrt{L}$ to $\sqrt{L'}$

Cor. Area ASP : area ASQ :: $\sqrt{L}$: $\sqrt{L'}$.

PROP. XV.

To determine the diameter of the circle of curvature at any point of a parabola, and the chord which passes through the focus.

DEF. If a curve PQ and a circle PqL touch the

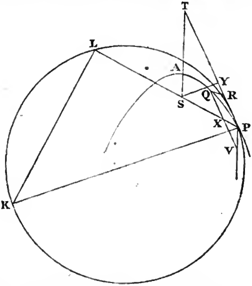


same straight line PR at the same point P, the circle is said to be a *Circle of Curvature* to the curve, when their deflections QR, qR from the common tangent PR are ultimately equal, or, which is the same thing, when indefinitely small arcs PQ, Pq, of the curve and circle, being equally deflected from the common tangent, are coincident.

COR. If any chord PL be drawn, and the subtense qQR parallel to PL, then the triangles PqL, PqR are equiangular (Eucl. 29. I. and 32. III.) Hence $PL = \frac{\overline{\text{chord. } Pq}|^2}{qR}$. But in the evanescent state of the figure PqR, the arcs PQ, Pq are coincident and equal, and the subtenses QR, qr: also the chord Pq is then equal to the arc Pq (NEWT. Lem. 7.)

Hence $PL = \frac{\overline{\text{arc } PQ}|^2}{QR}$, PQ and QR being taken in the evanescent state.

Let now PLK be the circle of curvature at any



point P of the parabola, PL the chord through the focus S, PK the diameter, QR parallel to PL, and QV parallel to PT cutting SP in X. Then $QR = PX = PV$, because $SP = ST$ (Prop. 4.). Also QV is ultimately equal to the arc PQ. (NEWT. Lem. 7.)

Hence $PL = \frac{PQ^2}{QR} = \frac{QV^2}{QR} = \frac{4SP \times PV}{QR} = 4SP$.

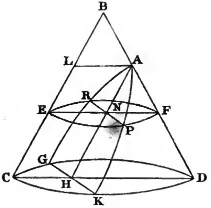
By sim. Δ^s SPY, SKL, the diameter $PK =$

$\frac{PL \times SP}{SY} = \frac{4SP^2}{SY}$, or $= \frac{4SP^2}{\sqrt{SP \times SA}}$ (Prop. 8.) $= \frac{4SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}}$,

or $= \frac{4SP^3 \times SA}{SY \times SP \times SA} = \frac{L \times SP^3}{SY^3}$.

PROP. XVI.

If a right cone BCD *be cut by a plane* AGK *which is parallel to a plane touching the slant side* BC, *the section* AGK *is a parabola.*



Let BCD be that position of the generating triangle, which is perpendicular to the cutting plane AGK; AH, their common section, which is parallel to BC (Eucl. 16. XI.) Draw AL parallel to CD. Then since the plane BCD passes through the axis, it is perpendicular to the base CKD and to every circular section EPF parallel to the base: it is also perpendicular to AGK. Hence the common section PR of the planes AGK, EPF is perpendicular to BCD and therefore to AH and EF (Euc. 19. XI.)

But $AN : NF :: BC : CD$, which is a constant ratio, therefore $AN \propto NF \propto EN \times NF$ (for EN is equal and parallel to AL, and constant) $\propto NP^2$ by the

property of the circle. Hence the curve is a parabola, whose axis is AH (Prop. 7.)

Cor. If L be the latus rectum of the parabola GAK, $L \times AN = NP^2 = EN \times NF$,

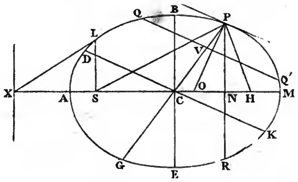
$$\therefore L = EN \times \frac{NF}{AN} = AL \times \frac{AL}{BL} = \frac{AL^2}{BL}.$$

ON

The Ellipse.

DEFINITIONS.

1. If two straight lines, *SP, HP,* revolve about



two fixed points *S, H,* intersecting in *P,* so that the sum of *SP* and *HP* may always be the same, the curve passing through all the points *P* is called an *Ellipse.*

2. The points *S, H,* are called the *Foci,* and the point *C* bisecting *SH,* the *Center.*

3. If *SH* be produced both ways to meet the curve in the points *A* and *M,* the line *AM* is called the *Axis Major;* and it's extremities *A, M,* are called *Vertices.*

4. A perpendicular PNR to the axis major, terminated by the curve, is called an *Ordinate* to the axis, and the segments AN, NM, into which it divides the axis, the *Abscissæ*. Also the ordinate BCE through the center is called the *Axis Minor*, and that through either focus, the *Latus Rectum*.

5. Any line PCG through the center is called a *Diameter;* and a diameter DCK drawn parallel to the tangent at the extremity P of PCG is called a *Conjugate Diameter* to PCG.

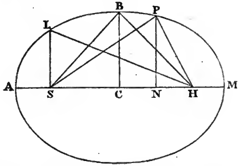
6. A line QQ' drawn parallel to the tangent at any point P is called an *Ordinate* to the point P or diameter PG; and the segments PV, VG, of the diameter, the *Abscissæ*.

7. A perpendicular PO to the tangent at any point P, terminated by the axis major, is called a *Normal;* and the part ON of the axis, the *Subnormal*.

8. A perpendicular to the axis major through that point X where a tangent at the extremity of the latus rectum meets the axis is called the *Directrix*.

PROP. I.

The sum of the focal distances SP, HP, *is equal to the axis major.*



$$\left.\begin{array}{l}\text{For } SP + HP = SA + HA \\ \text{And } SP + HP = SM + HM\end{array}\right\}\text{ (Def}^{\text{n}}\text{. 1.)}$$

$\therefore 2\,(SP + HP) = 2AM$, by addition.

COR. 1. $SH + 2SA = SH + 2HM$; $\therefore SA = HM$, and the center bisects the axis major.

COR. 2. $HP = 2AC - SP$.

PROP. II.

The distance of either focus from the extremity of the axis minor is equal to half the axis major.

For the $\triangle^s$ SCB, HCB are manifestly similar and equal. But $SB + BH\,(2SB) = 2AC$ (Prop. 1.)

COR. $AS \times SM = CA^2 - CS^2 = SB^2 - CS^2 = BC^2$.

PROP. III.

The latus rectum is a third proportional to the axis major and minor.

$HL^2 = (2AC - SL)^2$, and also $= HS^2 + SL^2$;

$\therefore 4AC^2 - 4AC \times SL + SL^2 = 4SC^2 + SL^2$;

Hence $AC \times SL = AC^2 - SC^2 = BC^2$

Or $AC : BC :: BC : SL$.

PROP. IV.

The focal distance $SP = \frac{BC^2}{AC - SC \times \cos. PSN}$, *radius being unity.*

$(2AC - SP)^2 = HP^2$

$= SP^2 + SH^2 - 2SH \times SN$ (E. 13. II.)

Or $4AC^2 - 4AC \times SP + SP^2 = SP^2 + 4SC^2 - 4SC.SN$

Hence $AC \times SP - SC \times SN = AC^2 - SC^2 = BC^2$.

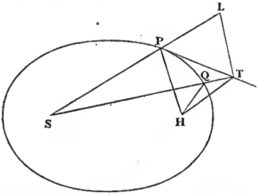
But $SN = SP \times \cos. PSN$, rad $= 1$,

$\therefore AC \times SP - SC \times SP \times \cos. PSN = BC^2$,

And $SP = \frac{BC^2}{AC - SC \times \cos. PSN}$.

PROP. V.

If one of the focal distances SP *be produced, the line* PT, *which bisects the exterior angle* HPL, *touches the curve at* P.



In SP produced take $PL = PH$, and in PT take any point T; join ST meeting the ellipse in Q: join also LT, HT, HQ.

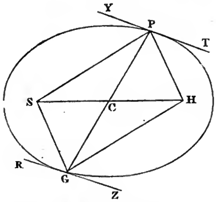
Then $LP = HP$, PT is common, and $\angle LPT = \angle HPT$, whence $LT = HT$; also $ST + HT = ST + TL$

and is therefore greater than SL, or than $SP+HP$, or than $SQ+HQ$. Hence the point Q lies within the $\triangle\ STH$ (Eucl. 21. I.), or T is without the ellipse; and since every point of PT, except P, is without the ellipse, PT touches at P.

Cor. 1. SP, HP make equal angles with the tangent PT.

Cor. 2. A tangent at the extremity of either axis is perpendicular to that axis.

Cor. 3. Complete the parallelogram $SPHG$.



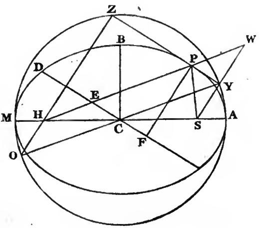
Then $SG+GH=SP+PH$ (Eucl. 34. I.), and G is a point in the ellipse. Join CP, CG; then because the diagonals of parallelograms bisect each other, and that $SC=CH$, PCG is a straight line and a diameter; *i. e.* the center bisects all diameters.

Cor. 4. Draw the tangents YT, RZ, at P, G: Then $\angle\ SPY = \angle\ HPT$ (Cor. 1.) and $\therefore = \frac{1}{2}$ supp^t. of $SPH = \frac{1}{2}$ supplement of $SGH = \angle\ HGZ$; and $\angle\ SPG =$ altern. $\angle\ PGH$; $\therefore\ \angle\ YPG = \angle\ PGZ$, and YT is parallel to RZ.

PROP. VI.

The perpendiculars from the foci upon any tangent intersect the tangent in the circumference of a circle whose diameter is the axis major.

Produce HP to W, making $PW = SP$; let SW cut the tangent in Y, and join CY.



The $\triangle$s SPY, WPY are equal in all respects (Eucl. 4. 1.), whence $\angle SYP = \angle WYP$, and SY is perpendicular to PY. Also $SY = YW$, and $CS = CH$, therefore CY is parallel to HW, and by sim. $\triangle$s, $CY = \frac{1}{2}\, HW = \frac{1}{2}\,(SP + HP) = CA$; therefore Y is a point in the circumference of the circle whose radius is CA.

Cor. 1. Let a conjugate diameter CD cut either focal distance HP in E. Then by the proposition, CY is parallel to HP, and $PECY$ is a parallelogram. Hence $PE = CY = AC$.

Cor. 2. Let HP coincide with the latus rectum; it is then perpendicular to AM: but CY is parallel to HP and therefore coincides with the axis minor; *i. e.* the axis minor produced meets a tangent at the extremity of the latus rectum in the circumference of the circle.

Prop. VII.

The rectangle by the perpendiculars from the foci upon the tangent is equal to the square of half the minor axis.

Produce ZH to the circumference in O; join CO. Then $\angle\ OZY$ being a right angle is in a semicircle, and O, Y, are the extremities of a diameter: OCY is therefore a straight line and a diameter; also the $\Delta^s\ OHC,\ YSC$ are similar and equal, and $SY = OH$; $\therefore SY \times HZ = OH \times HZ = AH \times HM$ (Eucl. 35. III.) $= BC^2$ (Cor. Prop. 2.)

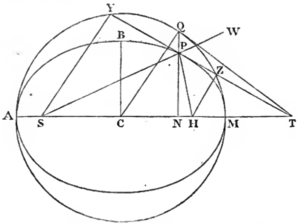
Cor. Since $\angle\ SPY = \angle\ HPZ$ (Prop. 5. Cor. 2.), the $\Delta^s\ SPY,\ HPZ$ are similar.

Hence $SY = HZ \times \dfrac{SP}{HP}$, and $SY^2 = BC^2 \times \dfrac{SP}{HP}$.

Also $SY^2 \propto \dfrac{SP}{HP}$ or $\propto \dfrac{SP}{2AC - SP}$.

PROP. VIII.

The semi-axis major is a mean proportional between the distance CN *of any ordinate from the center and the distance* CT *of the intersection of the tangent at* P *with the axis.*



The exterior $\angle\, WPH$ being bisected by PT,

$ST : HT :: SP : HP$ (Eucl. VI. Prop. A.)

$\therefore ST + HT : ST \sim HT :: SP + HP : SP \sim HP$

Or $2CT : SH :: 2AC : SP \sim HP.$

But $SH : SP + HP :: SP \sim HP : SN \mp HN$

Or $SH : 2AC :: SP \sim HP : 2CN;$

Hence $CT : CA :: CA : CN.$

COR. Produce NP to the circle in Q; join TQ, CQ. Then $CT = \dfrac{CA^2}{CN} = \dfrac{CQ^2}{CN}$; hence $\angle\, CQT$ is a right angle (Eucl. 8. VI.) and QT touches at Q; *i. e.* the tangents at P and Q intersect the axis in the same point T.

PROP. IX.

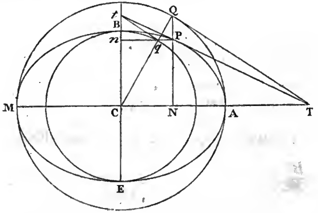
The rectangle by the abscissæ of the axis major is to the square of their semi-ordinate as the square of the axis major to the square of the axis minor.

The Δ^s PNT, STY, HTZ, being similar,

$$PN : SY :: TN : TY,$$
$$PN : HZ :: TN : TZ;$$
$$\therefore PN^2 : SY \times HZ :: TN^2 : TY \times TZ \; (TQ^2)$$
$$\text{Or } PN^2 : BC^2 :: QN^2 : CQ^2 \text{ by sim. } \Delta^s$$
$$:: AN \times NM : CA^2.$$

COR. 1. $PN : QN :: BC : AC.$

COR. 2. Let CQ cut the circle upon the axis



minor in q; join Pq, and produce it to the axis minor in n. Then because $PN : QN :: Cq : CQ$ (Cor. 1.), Pqn is parallel to CN and perpendicular to BC.

Hence $qn^2 : CN^2 :: Cn^2 \; (PN^2) : QN^2$ by sim. Δ^s

Or $Bn \times nE : Pn^2 :: BC^2 : AC^2$.

COR. 3. $Pn : qn :: AC : BC$

Cor. 4. Produce the tangent TP to meet the axis minor in t; then

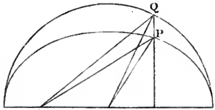
$Ct : Cn :: CT : NT :: CT^2 : QT^2$

$:: Cq^2$ or $BC^2 : Cn^2$

$\therefore Ct : BC :: BC : Cn$ as for the axis major.

Cor. 5. $AN \times NM$ or $CA^2 - CN^2 \propto NP^2$, and $Bn \times nE \propto Pn^2$.

Cor. 6. As in the Parabola, Prop. xiv, the



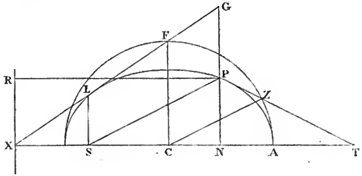
area ANP of the ellipse : the area ANQ of the circle :: $NP : NQ$, *i. e.* :: $BC : AC$. The whole areas therefore of the ellipse and circle are in the same proportion. Hence the area of the ellipse = area of the circle $\times \dfrac{BC}{AC}$ which for different ellipses $\propto AC^2 \times \dfrac{BC}{AC} \propto AC \times BC \propto$ the rectangle by the axes.

Cor. 7. The areas of circles being as the squares of their diameters, the area of an ellipse = the area of that circle whose diameter is a mean proportional between the axes.

Cor. 8. $\left.\begin{array}{l}\text{Area } ASP : \text{area } ASQ \\ \text{and area } ACP : \text{area } ACQ\end{array}\right\} :: BC : AC.$

Prop. X.

A semi-ordinate NP *produced to meet the tangent at the extremity of the latus rectum is equal to the focal distance* SP.



The intersection F of the axis minor with the tangent at L is in the circumference of the circle (Prop. 6. Cor. 2.)

By sim. $\triangle^s$ $NG : CF :: NX\,(CX \pm CN) : CX,$

$$:: \frac{CA^2}{CS} \pm \frac{CA^2}{CT} : \frac{CA^2}{CS} \text{ (Prop. 8.)}$$

$$:: CT \pm CS\,(TS) : CT$$

$$:: SP : CZ \text{ (Prop. 6.)}$$

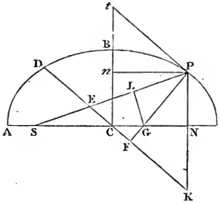
$\therefore NG = SP.$

Cor. Draw PR perpendicular to the directrix XR. By sim. $\triangle^s$ $NG\,(SP) : NX\,(PR) :: CF\,(CA) : CX$

$$:: CS : CA.$$

Prop. XI.

The rectangle by the normal PG, *and the perpendicular* PGF *upon the conjugate diameter is equal to the square of half the minor axis.*

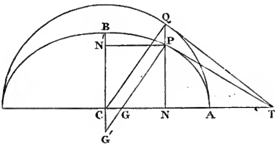


Let PN produced meet the conjugate in K. Then the angles N and F being right angles, a circle would circumscribe the quadrilateral $FGNK$; hence $PG \times PF = PN \times PK$ (Eucl. 36. III.) $= Cn \times Ct$ $= BC^2$ (Prop. 9. Cor. 4.)

Cor. From G draw GL perpendicular to SP; then from the quadrilateral $EFGL$, whose opposite angles F and L are right angles, $PL = \dfrac{PF \times PG}{PE} = \dfrac{BC^2}{AC} =$ half the latus rectum (Prop. 3.).

Prop. XII.

The subnormal $NG = \frac{L}{2AC} \times CN$, L *being the latus rectum.*



For $NG : CN :: \frac{NP^2}{NT} : \frac{NQ^2}{NT}$ (Eucl. 8. vi.)

$:: BC^2 : AC^2$ (Prop. 9. Cor. 1.)

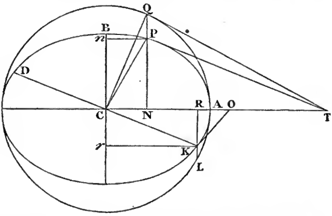
$:: L : 2AC$ (Prop. 3.)

Similarly for the axis minor, $N'G' = \frac{2AC}{L} \times CN'$.

Prop. XIII.

If one diameter DCK *be conjugate to another* PC, *conversely* PC *is conjugate to* DCK.

Draw PN, KR, ordinates to the axis, and the tan-



gents PT, KO. Then the Δ^s PNT, CKR are similar;
But $CN \times NT = NQ^2 = AN \times NM$ (Eucl. 8. vi.)
$\therefore CN \times NT : CR \times RO :: PN^2 : KR^2$ (Prop. 9. Cor. 5.)
$:: NT^2 : CR^2$

Hence $CN : RO :: NT : CR :: PN : KR$;
$\therefore$ Δ^s CPN, KRO are similar (Eucl. 6. vi.) and PCG is parallel to KO.

Prop. XIV.

The distance of one of the ordinates PN, KR *from the center is a mean proportional between the distances of the other ordinate from the center and from the intersection of it's tangent with the axis.*

$CN \times CT = CR \times CO$, for each $= CA^2$ (Prop. 8.)
Hence $CN : CR :: CO : CT$
$:: CK : PT$ by sim. Δ^s CPT, CKO
$:: CR : NT$ by sim. Δ^s CKR, PNT.
So $CR : CN :: CN : RO$.

Cor. 1. $CR^2 = CN \times NT = NQ^2$ or $= AN \times NM$.
So $CN^2 = AR \times RM$, and $CN = RL$.

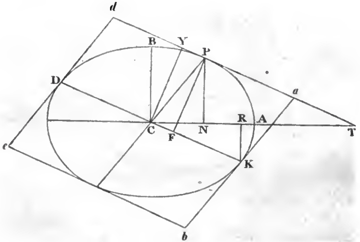
Cor. 2. $CN^2 + CR^2 = CN^2 + NQ^2 = CA^2$. By a similar proof, if Pn, Kr be perpendicular to the axis minor, $Cn^2 + Cr^2 = BC^2$.

Cor. 3. $AC^2 + BC^2 = CN^2 + Cn^2 + CR^2 + Cr^2 = CP^2 + CD^2$.

Cor. 4. $NP : CR :: BC : AC$ (Prop. 9. Cor. 1.)
So $KR : CN :: BC : AC$.

Prop. XV.

All parallelograms circumscribing an ellipse are equal.



If tangents be drawn at the extremities of two conjugate diameters, they form a parallelogram $abcd$ circumscribing the ellipse (Prop. 5. Cor. 4. and Prop. 13.)

of which $CPaK$ is a fourth part. Draw CY perpendicular to PT; then by sim. Δ^s,

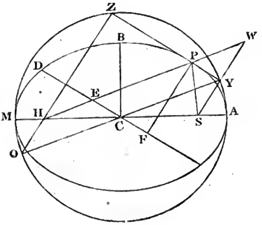
$$CT\left(\frac{CA^2}{CN}\right) : CY\,(PF) :: CK\,(CD) : KR$$

Also $BC : AC :: KR : CN$ (Prop. 14. Cor. 4.)

Hence $CD \times PF = AC \times BC$, or the parallelogram PK=the rectangle by the semi-axes.

Cor. 1. The whole area of the ellipse, which varies as $AC \times BC$ (Prop. 9. Cor. 6.), varies also as $CD \times PF$, or as the circumscribing parallelogram.

Cor. 2. Draw the perpendiculars from the foci upon the tangent at P.



$$\left.\begin{array}{l}\text{Then } SP : SY \\ \text{And } HP : HZ\end{array}\right\} :: PE\,(AC) : PF,$$

$\therefore SP \times HP : SY \times HZ :: AC^2 : PF^2$

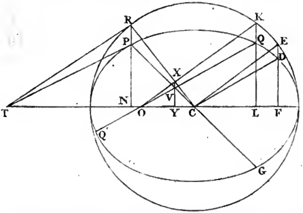
$:: CD^2 : BC^2$ by the Prop.

But $SY \times HZ = BC^2$; (Prop. 7.) $\therefore SP \times HP = CD^2$.

Prop. XVI.

The rectangle by the abscissæ PV, VG, *of any diameter is to the square of their semi-ordinate* QV *as the square of the semi-diameter* CP *to the square of the semi-conjugate* CD.

Let the ordinate QQ' meet the axis in O: draw



RPN, VY, QL, EDF, perpendicular to the axis: let YV meet CR in X, and OX meet LQ in K. Then since $KL : QL :: XY : VY :: RN : PN$, by similar triangles, K is in the circumference of the circle (Prop. 9. Cor. 1.) But CD is parallel to PT; $\therefore\ TN : CF :: PN : DF :: RN : EF$; whence the $\triangle^s$ TNR, CFE, are similar, and EC is parallel to RT and perpendicular to CR. In like manner KO is perpendicular to CR.

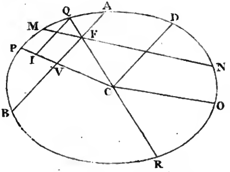
$$\begin{array}{llll} \text{Now} & CP^2 : CV^2 & :: & CR^2 : CX^2 \\ \therefore & CP^2 - CV^2 : CR^2 - CX^2\ (KX^2) & :: & CP^2 : CR^2 \\ \text{And} & KX^2 : QV^2 & :: & CE^2 : CD^2 \\ \therefore & CP^2 - CV^2\ (PV \times VG) : QV^2 & :: & CP^2 : CD^2. \end{array}$$

Cor. 1. $PV \times VG : Q'V^2 :: CP^2 :: CD^2$, by the same proof; $\therefore QV = Q'V$; *i. e.* a diameter bisects all its own ordinates.

Cor. 2. $PV \times VG$ or $CP^2 - CV^2 \propto QV^2$.

Prop. XVII.

If any line AB *intersect a diameter* QCR *in* F, *and* CD *be parallel to* AB, *then* AF × FB : QF × FR :: CD² : CQ².



Draw CP conjugate to CD and therefore bisecting AB in V. Also draw QI parallel to AB; then

$AV^2 : QI^2 :: CP^2 - CV^2 : CP^2 - CI^2$ (Pr. 16. Cor. 2.)

$QI^2 : FV^2 :: CI^2 : CV^2$ by sim. Δ^s.

$\therefore AV^2 : FV^2 :: CI^2.(CP^2 - CV^2) : CV^2.(CP^2 - CI^2)$

Div^{do} $AV^2 : AV^2 - FV^2 :: CI^2 (CP^2 - CV^2) : CP^2.(CI^2 - CV^2)$

And $CP^2 - CV^2 : AV^2 :: CP^2 : CD^2$

$\therefore AV^2 - FV^2 : CD^2 :: CI^2 - CV^2 : CI^2$

$:: CQ^2 - CF^2 : CQ^2$ by sim. Δ^s.

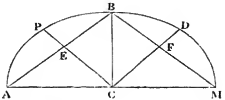
Or $AF \times FB : CD^2 :: QF \times FR : CQ^2$.

Cor. 1. If any two lines whatever AB, MN intersect in F, the rectangles $AF \times FB$, $MF \times FN$, are as the squares of the diameters CD, CO, drawn parallel to AB, MN, respectively.

Cor. 2. If AB, MN, move parallel to themselves, and intersect any where either within or without the ellipse in F, the rectangles $AF \times FB$, $MF \times FN$, have always the same constant ratio.

Prop. XVIII.

The diameters which bisect the lines joining the extremities of the axes are equal and conjugate.

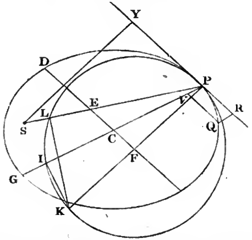


Let AB, BM be bisected in E, F: then the $\triangle^s$ BCE, BCF are manifestly equal and similar, and $\angle\ ECB = \angle\ FCB$; therefore CP and CD making equal angles with CB are similarly drawn in the two elliptic quadrants and consequently are equal. Also AB is bisected by CP and is therefore an ordinate belonging to the diameter CP (Prop. 16. Cor. 1.) But $AC = CM$, and $BF = FM$; hence CFD is parallel to AB or is conjugate to CP.

Prop. XIX.

To determine the diameter of curvature and the chords through the center and focus of an ellipse.

I. The chord PI through the center.



Draw QR parallel to PI and QV parallel to the tangent; then when PQ is evanescent, $PQ = QV$.

Now $PV \times VG : QV^2 :: CP^2 : CD^2$ (Prop. 16.); hence $\frac{QV^2}{PV} = VG \times \frac{CD^2}{CP^2}$; and ultimately $\frac{PQ^2}{QR} = \frac{2CP \times CD^2}{CP^2} = \frac{2CD^2}{CP} = PI$.

II. The diameter PK.

In the quadrilateral $CFKI$, the opposite angles F and I are right angles, and therefore a circle would circumscribe it. Hence $PK \times PF = CP \times PI = 2CD^2$, by the preceding case, and $PK = \frac{2CD^2}{PF}$.

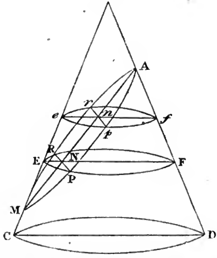
III. The chord PL through the focus.

Let it cut the conjugate in E; then $PE =$ the semi-axis major. Also, as in the last case, $PL \times PE = PK \times PF = 2CD^2$; $\therefore PL = \frac{2CD^2}{PE}$.

Cor. The diameter $PK = \frac{2CD^2}{PF} = \frac{2PE^2 \times BC^2}{PF^3}$ (Prop. 15.) $= \frac{2BC^2}{PE} \times \frac{PE^3}{PF^3} = L \times \frac{SP^3}{SY^3}$ by similar triangles.

Prop. XX.

If a right cone BCD *be cut by a plane* ARMP *through both slant sides, the section is an ellipse.*



Let BCD be that position of the generating triangle which is perpendicular to the cutting plane; EPF, epf, any circular sections. Then, as in the Parabola, Prop. XVI, the common sections PNR, pnr, are perpendicular to AM and to EF, ef, respectively.

Now $AN : NF :: An : nf$

And $NM : EN :: nM : en.$

$\therefore AN \times NM : EN \times NF :: An \times nM : en \times nf$

Or $AN \times NM : NP^2 :: An \times nM : np^2$

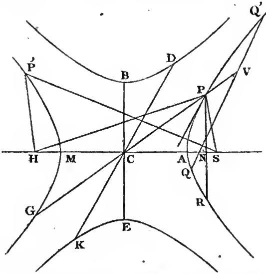
i. e. the rectangle by the abscissæ varies as the square of the corresponding semi-ordinate, a property of the ellipse (Prop. 9. Cor. 5.)

ON

The Hyperbola.

DEFINITIONS.

1. If two straight lines SP, HP, revolve about two fixed points S, H, and intersect in P so that the



excess of HP above SP may always be the same, the curve PAR passing through all the points P is called an *Hyperbola*.

Cor. If $HP - SP$ and $SP' - HP'$ be always equal to the same constant quantity, the points P, P' will lie in two opposite and similar hyperbolas PA, $P'M$.

2. The points S, H, are called the *Foci*; the bisection C of the line SH joining the foci is called the *Center*, and the part AM of that line intercepted between the two opposite hyperbolas, the *Axis Major*.

3. If a perpendicular BE to the axis major drawn through the center be cut in the points B, E by a circle whose center is either extremity of the axis major, and whose radius is half the interval between the foci, BE is called the *Axis Minor* or *Conjugate Axis*.

4. The two opposite hyperbolas BD, EK, whose axis major is BE and conjugate axis AM, are called the *Conjugate Hyperbolas* to AP, MP'.

Cor. The distance of the foci of BD, EK, from the center is equal to SC.

5. If the axes AM, BE be equal, the curve is called a *Rectangular Hyperbola*. In this case the conjugate hyperbolas AP, BD, are equal and similar.

6. Any line PCG drawn through the center and terminated by two opposite hyperbolas is called a *Diameter*, and the intersections of a diameter with the curves are called it's *Vertices*.

7. A perpendicular PNR to the axis major, terminated by the curve, is called an *Ordinate* to the axis; and the distances AN, NM of the ordinate from the vertices are called the *Abscissæ*.

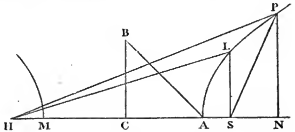
8. A line QQ' drawn parallel to the tangent at any point P and terminated by the curve, is called an *Ordinate* to the point P or diameter PCG; and the distances PV, VG of the ordinate from the vertices of the diameter are called the *Abscissæ*.

9. The *Latus Rectum, Directrix,* and *Conjugate Diameter* are defined as in the Ellipse.

10. An *Asymptote* is a straight line which approaches nearer to meet a curve, the farther it is produced, but which being produced ever so far does never actually meet it.

Prop. I.

The difference of the focal distances SP, HP, *is equal to the axis major.*



For $HP - SP = HA - SA$
And $HP - SP = SM - HM$
$\therefore 2\,(HP - SP) = 2AM$, by addition.

Cor. 1. $SH - 2SA = SH - 2HM$; $\because SA = HM$, and the center C bisects the axis major.

Cor. 2. $HP = 2AC + SP$.

Cor. 3. The rectangle $MS \times SA = SC^2 - AC^2$ (Eucl. 6. II.) $= AB^2 - AC^2 = BC^2$ (Def. 3).

Prop. II.

The latus rectum is a third proportional to the axis major and minor.

For $HS^2 + SL^2 = HL^2 = (2AC + SL)^2$
Or $4SC^2 + SL^2 = 4AC^2 + 4AC \times SL + SL^2$.
Take away SL^2, and $AC \times SL = SC^2 - AC^2 = BC^2$.
Or $AC : BC :: BC : SL$.

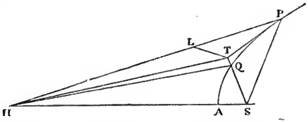
Prop. III.

The focal distance $SP = \frac{BC^2}{AC - SC \times \cos. PSN}$, *radius being unity.*

For since $(2AC+SP)^2 = HP^2$,
$4AC^2 + 4AC \times SP + SP^2 = SP^2 + SH^2 + 2SH \times SN$
Whence $AC \times SP - SC \times SN = SC^2 - AC^2 = BC^2$.
Or, since $SN = SP \times \cos. PSN$,
$AC \times SP - SC \times SP \times \cos. PSN = BC^2$,
And $SP = \frac{BC^2}{AC - SC \times \cos. PSN}$.

Prop. IV.

The line PT, *which bisects the angle* SPH, *touches the hyperbola at* P.



In PH take $PL = SP$, and in PT take any point T; join ST, and let it meet the curve in Q; join also LT, HT, HQ. Then $LT = ST$ (Eucl. 4. 1.) But the difference of any two sides of a triangle is less than the third side; therefore $HT - LT$ or $HT - ST$ is less than HL or $HP - SP$ or $HQ - SQ$. Hence

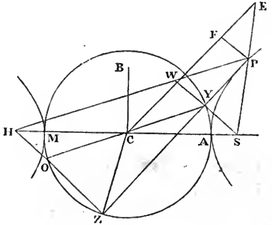
ST is greater than *SQ*; for since $HT - ST$ is less than $HQ - SQ$, if *ST* were less than *SQ*, $HT + TQ$ would be less than *HQ*, which is impossible. Therefore every point *T* in *PT*, except *P*, is without the hyperbola, and *PT* touches at *P*.

Cor. 1. *SP* and *HP* make equal angles with every tangent *PT*.

Cor. 2. A tangent at the vertex of either axis is perpendicular to that axis.

Prop. V.

The perpendiculars from the foci on any tangent intersect the tangent in the circumference of a circle whose diameter is the axis major.



In *PH* take $PW = SP$, and then $HW = AM$; let *SW* cut the tangent in *Y* and join *CY*.

Then the triangles SYP, WYP are equal in all respects; $\therefore\ \angle\, SYP = \angle\, WYP$, and SY is perpendicular to PY. Also $SY = YW$, and $SC = CH$; $\therefore\ CY$ is parallel to HW, and $= \frac{1}{2} HW = CA$; *i. e.* Y is a point in the circumference.

Cor. 1. Join CZ which is parallel to SP, as in the proposition; and let SP produced meet the conjugate diameter in E. Then $PECZ$ is a parallelogram, and $PE = CZ = AC$.

Cor. 2. If SP coincide with the latus rectum, CZ coincides with the axis minor: therefore the axis minor meets the tangent at the extremity of the latus rectum in the circumference of the circle.

Prop. VI.

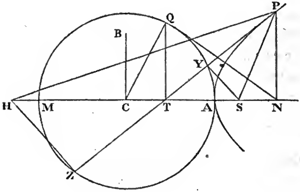
The rectangle by the perpendiculars from the foci on any tangent is equal to the square of half the minor axis.

Let HZ meet the circle again in O, and join CO; then the right angle YZO is in a semicircle; hence YCO is a diameter and a straight line; also the triangles CSY, CHO are equal in all respects; hence $HO = SY$, and $SY \times HZ = HO \times HZ = HA \times HM$ (Eucl. 36. III.) $= BC^2$ (Prop. 1. Cor. 3.).

Cor. $SY^2 = BC^2 \times \dfrac{SP}{HP}$, as in the Ellipse, and $SY^2 \propto \dfrac{SP}{HP} \propto \dfrac{SP}{2AC + SP}$.

Prop. VII.

The semi-axis major is a mean proportional between the distance CN *of any ordinate to the axis from the center and the distance* CT *of the intersection of the axis with the tangent at the extremity of the ordinate.*



Draw TQ perpendicular to the axis; join NQ, CQ.

Then $HT : ST :: HP : SP$ (Eucl. 3. vi.)

$\therefore HT - ST : HT + ST :: HP - SP : HP + SP$

Or $2CT : SH :: 2AC : HP + SP$

But $SH : HP + SP :: HP - SP : HN \mp SN$

$:: 2AC : 2CN;$

Hence $CT : CA :: CA : CN.$

Cor. $CN = \dfrac{CA^2}{CT} = \dfrac{CQ^2}{CT}$; whence $\angle\ CQN$ is a right angle; and NQ touches the circle at Q.

Prop. VIII.

The rectangle by the abscissæ of the axis major is to the square of their semi-ordinate as the square of the axis major to the square of the axis minor.

The triangles PTN, STY, HTZ, being similar,

$PN : SY :: TN : TY$

$PN : HZ :: TN : TZ$

$\therefore PN^2 : SY \times HZ \; (BC^2) :: TN^2 : TY \times TZ \; (QT^2)$

$:: NQ^2 : CQ^2$

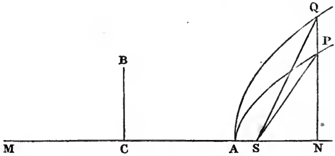
$:: AN \times NM : CA^2.$

Cor. 1. $PN^2 : CN^2 - CA^2 :: BC^2 : AC^2.$

Cor. 2. $PN^2 \propto AN \times NM \propto CN^2 - CA^2.$

Cor. 3. In the rectangular hyperbola, the rectangle of the abscissæ is equal to the square of the corresponding semi-ordinate. (Def. 5.)

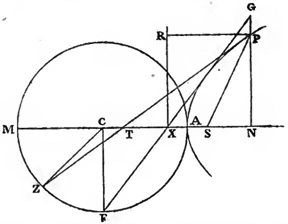
Cor. 4. If AQ be a rectangular hyperbola, AP any



other hyperbola having the same axis major, and QPN a common ordinate of the axis, then $PN : QN :: BC : AC$. Hence, as in the Parabola, Prop. xiv, the areas APN, AQN, as also the areas ASP, ASQ, are in the same constant proportion of BC to AC.

Prop. IX.

A semi-ordinate NP *of the axis being produced to meet the tangent at the extremity of the latus rectum is equal to the focal distance* SP.



The axis minor meets the tangent GF at the extremity of the latus rectum in the circumference of the circle, and CZ is parallel to SP. (Prop. 5. Cor. 2.)
By sim. $\triangle^s$ $NG : CF :: NX\ (CN - CX) : CX$

$$:: \frac{CA^2}{CT} - \frac{CA^2}{CS} : \frac{CA^2}{CS}$$

$$:: CS - CT\ (ST) : CT$$

$$:: SP : CZ$$

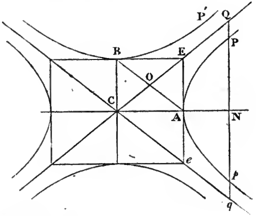
$\therefore NG = SP$.

Cor. Draw PR perpendicular to the directrix XR.
Then $NG\ (SP) : NX\ (PR) :: CF\ (CA) : CX$

$$:: CS : CA.$$

Prop. X.

If tangents be drawn at the vertices of the axes, the diagonals of the rectangle so formed are asymptotes to the four curves.



Let NP meet CE in Q.
Then $NQ^2 : CN^2 :: AE^2\ (BC^2) : AC^2$
$:: NP^2 : CN^2 - CA^2$ (Pr. 8. Cor. 1.)
Now as CN increases, the ratio of CN^2 to $CN^2 - CA^2$ continually approaches to equality; but $CN^2 - CA^2$ is never actually equal to CN^2, if CN be ever so much increased. Hence NP is always less than NQ, but approaches continually nearer to equality with it.

By the same proof, CQ is an asymptote to the conjugate hyperbola BP'.

Cor. 1. The two asymptotes make equal angles with the axis major, and with the axis minor.

Cor. 2. The line AB joining the vertices of the conjugate axes is bisected by one asymptote and parallel to the other.

It is bisected by CQ because the diagonals of the rectangle $BCAE$ bisect each other; and it is parallel to Ce, because $EO = OC$ and $EA = Ae$ (Eucl. 2. vi.)

Cor. 3. All lines perpendicular to either axis and terminated by the asymptotes are bisected by the axis.

Cor. 4. In the rectangular hyperbola, the asymptotes are at right angles to each other.

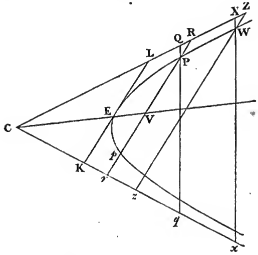
Prop. XI.

If any line Qq *perpendicular to either axis be terminated by the asymptotes, the rectangle of the segments, into which the curve divides it, is equal to the square of half the conjugate axis.*

For $NQ^2 : CN^2 :: NP^2 : CN^2 - CA^2$ (Prop. 10.)
$\therefore NQ^2 - NP^2 : CA^2 :: NP^2 : CN^2 - CA^2$
$:: BC^2 : CA^2$ (Prop. 8. Cor. 1.)
Hence $NQ^2 - NP^2$ or $QP \times Pq = BC^2$, and is therefore the same wherever in the curve the point P is taken.

Prop. XII.

If any line whatever Rr, *making a given angle with either asymptote, cut the curve in* P, *the rectangle by the segments* RP, Pr, *is invariable.*



Draw any other line Zz parallel to Rr, and through P, W, draw Qq, Xx perpendicular to the axis. Then, by similar triangles,

$RP : QP :: ZW : XW$

$Pr : Pq :: Wz : Wx$,

$\therefore RP \times Pr : QP \times Pq :: ZW \times Wz : XW \times Wx$

But $QP \times Pq = XW \times Wx$ (Prop. 11.)

$\therefore \quad RP \times Pr = ZW \times Wz$.

Cor. 1. Let Rr move parallel to itself until the points P, p, coincide, as at E. Then LEK touches at E, and $RP \times Pr = LE \times EK$. Also by the same proof $Rp \times pr = LE \times EK$.

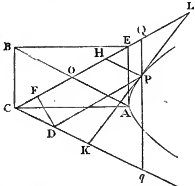
Cor. 2. $RP \times Pr = Rp \times pr$, or $RP \times Pp + RP \times pr = RP \times pr + Pp \times pr$ (Eucl. I. II.): whence $RP = pr$.

Cor. 3. $LE = EK$, and $RP \times Pr = LE^2$.

Cor. 4. Let the diameter CE cut Rr in V: then $VR = Vr$, by similar triangles, and $RP = pr$; whence $PV = Vp$; *i. e.* a diameter bisects all it's own ordinates.

Prop. XIII.

If from any point P *in the curve straight lines* PD, PH, *be drawn parallel to the asymptotes, their rectangle is invariable.*



Draw the tangent LPK, and Qq perpendicular to the axis. Then, by similar triangles,

$PH : PQ :: AO : AE$

$PD : Pq :: OE : AE$

$\therefore PD \times PH : PQ \times Pq :: AO^2 : AE^2$

But $PQ \times Pq = AE^2$ (Prop. 11.) $\therefore PD \times PH = AO^2 = \frac{1}{4}(AC^2 + BC^2)$ a constant quantity.

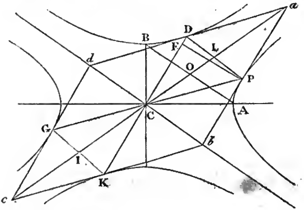
Cor. 1. Draw DF perpendicular to CH; then the triangle DCF is always similar to itself. Hence $CD \times CH$ and $DF \times CH$ are in a constant ratio; and therefore $DF \times CH$ or the parallelogram DH is a constant quantity.

Cor. 2. The triangle CLK is a constant quantity, being double of the parallelogram DH.

Cor. 3. $PH \propto \frac{1}{CH}$.

Prop. XIV.

If a straight line PD *be drawn parallel to one asymptote, and terminated by the conjugate hyperbolas, it is bisected by the other asymptote.*



For $CL \times LP = AO^2$ (Prop. 13.) $= BO^2 = CL \times LD$; $\therefore PL = LD$.

Cor. 1. If ab touch at P, it is bisected in P, and therefore $Ca = 2CL$ (Eucl. 2. vi.). For this reason a tangent dD at D must meet Ca in the same point a.

Also since $PL = LD$, $Cb = Cd$, by similar triangles; and Cb or $Cd = 2LP = PD$. Hence CD is parallel to ab, and CP to ad (Eucl. 33. I.).

Cor. 2. Draw bKc touching at K, and from c draw cGd touching at G. Then cb, cd being bisected at K, G, GK is parallel to bd and bisected in I. Hence, as in the last Cor^y, $Cd = Cb$, and ad, cd, meet Cd in the same point d. Also $IK = \frac{1}{2} Cb = LD$, whence $CI = CL$ (Prop. 13. Cor. 3.). The triangles CIK, CLD are therefore equal in all respects, and in like manner the triangles CLP, CIG. Hence PCG, DCK are straight lines and conjugate diameters; they are likewise bisected by the center C.

Cor. 3. The tangent ad is equal and parallel to the diameter PG, and ab to the conjugate DK.

Cor. 4. The figure $abcd$ is a parallelogram, of which $CPaD$ is a fourth part.

Prop. XV.

The parallelograms formed by tangents at the vertices of any pair of conjugate diameters have all the same area.

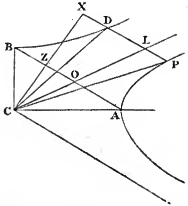
For the parallelogram $CPaD = 2 \triangle CPa = \triangle Cab$ (Eucl. 38. I.) a constant quantity (Prop. 13. Cor. 2.).

Cor. 1. Draw PF perpendicular to CD; then the parallelogram $CPaD =$ the parallelogram Db (Eucl. 35. I.) $= CD \times PF$. Now when the tangents are drawn at A and B, CD coincides with CB, and PF with AC. Hence $CD \times PF = AC \times BC$.

Cor. 2. As in the Ellipse, Prop. 15. Cor. 2, the rectangle by the focal distances is equal to the square of the semi-conjugate diameter.

Prop. XVI.

The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes.



Draw CZX perpendicular to AB and PD,

Then $CP^2 - CD^2 = PX^2 - DX^2$

$= 4PL \times LX$ (Eucl. 8. II.)

But $LX : OZ :: CL : CO$, by sim. Δ^s.

$:: AO : PL$ (Prop. 13. Cor. 3.)

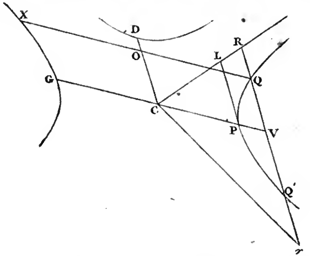
$\therefore 4PL \times LX \, (CP^2 - CD^2) = 4AO \times OZ$

$= AC^2 - BC^2$.

Prop. XVII.

The rectangle by the abscissæ PV, VG, *of any diameter* PG *is to the square of their semi-ordinate* QV *as the square of* CP *to the square of the semi-conjugate* CD.

Let QQ' meet the asymptotes in R, r. Then



$RQ \times Qr$ or $RV^2 - QV^2 = PL^2$ (Prop. 12. Cor. 3.) $= CD^2$ (Prop. 14. Cor. 3.) $\therefore QV^2 = RV^2 - PL^2$;

But $CV^2 : CP^2 :: RV^2 : PL^2$

Div^{do}. $CV^2 - CP^2 : RV^2 - PL^2 :: CP^2 : PL^2$

Or $PV \times VG : QV^2 :: CP^2 : CD^2$.

Cor. 1. $QV^2 \propto PV \times VG \propto CV^2 - CP^2$.

Cor. 2. Let QOX, parallel to PG, cut CD in O and the opposite hyperbola in X.

Then $CV^2 - CP^2 : CP^2 :: QV^2 : CD^2$

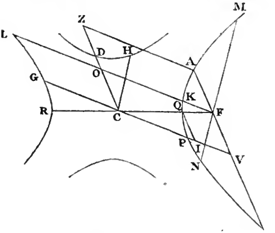
$$\left.\begin{array}{l} \text{Comp}^{do}.\ CV^2 : CD^2 + QV^2 \\ \text{Or} \quad QO^2 : CD^2 + CO^2 \\ \text{And so } XO^2 : CD^2 + CO^2 \end{array}\right\} :: CP^2 : CD^2,$$

Hence $QO = OX$; *i. e.* every line terminated by two opposite hyperbolas is bisected by that diameter to whose conjugate it is parallel.

Cor. 3. $QO^2 \propto CD^2 + CO^2$.

Prop. XVIII.

If any line AB *intersect a diameter* RQ *produced in* F, *and* CD *be the semi-diameter parallel to* AB, *then* $AF \times FB : QF \times FR :: CD^2 : CQ^2$.



Let CPV be the diameter bisecting AB, and draw QI parallel to AB; then (Prop. 17. Cor. 1.)

$$AV^2 : QI^2 :: CV^2 - CP^2 : CI^2 - CP^2,$$

And $QI^2 : FV^2 :: CI^2 : CV^2$ by sim. $\triangle^s$;

$$\therefore\ AV^2 : FV^2 :: CI^2 \times (CV^2 - CP^2) : CV^2 \times (CI^2 - CP^2)$$

Div^{do} $AV^2 : AV^2 - FV^2 :: CI^2 \times (CV^2 - CP^2) : CP^2 \times (CV^2 - CI^2)$;

But $CV^2 - CP^2 : AV^2 :: CP^2 : CD^2$ (Prop. 17.);

$$\therefore\ AV^2 - FV^2 : CD^2 :: CV^2 - CI^2 : CI^2$$

$$:: CF^2 - CQ^2 : CQ^2, \text{ by sim. } \triangle^s;$$

Or $AF \times FB : CD^2 :: QF \times FR : CQ^2$.

Cor. 1. If any other line MN cut AB in F, and CH be the semi-diameter parallel to MN, then $AF \times FB : MF \times FN :: CD^2 : CH^2$.

Cor. 2. Draw FKL parallel to VG, and cutting the opposite hyperbolas; also draw AZ parallel to VG: then (Prop. 17. Cor[s].)

$AZ^2\ (FO^2) : KO^2 :: CD^2 + CZ^2 : CD^2 + CO^2$

Div[do]. $FO^2 - KO^2 : CZ^2 - CO^2 :: KO^2 : CD^2 + CO^2$

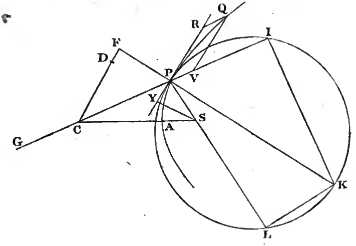
Or $KF \times FL : AF \times FB :: CP^2 : CD^2$

$\therefore\ KF \times FL : MF \times FN :: CP^2 : CH^2$.

Cor. 3. If AB, KL, MN, move parallel to themselves, and intersect in F either within or without the hyperbola, the rectangles $AF \times FB$, $KF \times FL$, $MF \times FN$, have always the same ratio to each other as the squares of the diameters to which the lines are respectively parallel.

Prop. XIX.

To determine the diameter of curvature and the chords through the center and focus of an hyperbola.



As in the Ellipse, Prop. 19,

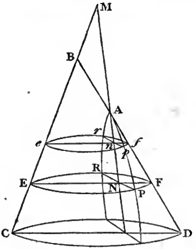
I. The chord PI through the center $= \frac{2CD^2}{CP}$.

II. The diameter $PK = \frac{2CD^2}{PF}$, or $= L \times \frac{SP^3}{SY^3}$

III. The chord PL through either focus $= \frac{2CD^2}{AC}$.

Prop. XX.

If a right cone BCD *be cut through one side* BD *by a plane* APR *which produced backwards cuts the other side* CB *produced, the section is an hyperbola.*



As in the Ellipse, Prop. 20, $AN \times NM \propto NP^2$, which is the property of an hyperbola whose axis major is AM (Prop. 8, Cor. 2.).

ON

THE SECTIONS

OF

The Conoids.

DEFINITIONS.

1. THE solids generated by the revolution of conic sections about their axes are called *Conoids*.

2. If a Parabola so revolve, the solid is called a *Paraboloid*.

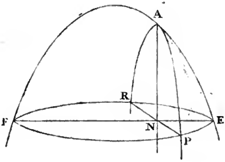
3. If an Ellipse revolve about it's axis major, the solid is called a *Prolate Spheroid;* and if it revolve about the axis minor, an *Oblate Spheroid*.

4. The solid generated by the revolution of an Hyperbola about it's axis major is called an *Hyperboloid*.

SCHOLIUM. Every ordinate to the axis of revolution describes a circle whose center is in the axis; therefore all sections, whose planes are perpendicular to the axis, are circles. Also the revolving plane, since it always passes through the axis, is perpendicular in every position to the planes of all the circular sections.

Prop. I.

If a paraboloid be cut by a plane parallel to the generating plane, the section is the same as the parabola which revolves.

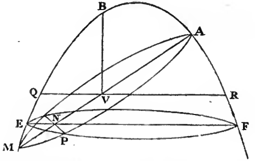


Let FAE be that position of the generating parabola which is perpendicular to the cutting plane RAP, and $FPER$ any circular section. Then both the planes FPE, RAP are perpendicular to FAE and therefore PNR is perpendicular to FAE and to the lines AN, EF. Also AN is parallel to the axis of the solid.

If L be the latus rectum of the revolving parabola, $L \times AN = EN \times NF$ (Parabola, Prop. 9.) $= NP^2$. Wherefore the curve RAP is the parabola whose latus rectum is L.

Prop. II.

Any other section of a paraboloid is an ellipse.



Let the generating plane MBF be perpendicular to the cutting plane AMP, AM their common section, which cutting the parabola in one point A, must cut it again, as at M (Parabola, Prop. 2. Cor. 1.). Also let BV be the diameter to which AM belongs, and draw QVR an ordinate of the axis.

Then (Parabola, Prop. 12. Cor. 2.)

$$AN \times NM : EN \times NF :: AV \times VM : QV \times VR$$

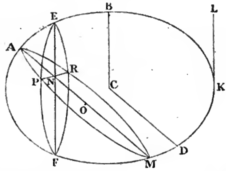
$$\text{Or } AN \times NM : NP^2 :: AV^2 : L \times BV.$$

The curve is therefore an ellipse, whose major axis is AM, and whose semi-axis-minor is a mean proportional between L and BV.

Prop. III.

All sections of a spheroid, except those which are perpendicular to the axis, are ellipses.

Let ABD represent a *prolate* spheroid, CB the



semi-axis minor of the revolving ellipse, CD a semi-diameter parallel to AM: then (Ellipse, Pr. 17. Cor. 1.)

$$AN \times NM : EN \times NF (NP^2) :: CD^2 : CB^2.$$

The curve is therefore an ellipse, whose axis major is AM, and whose axis minor is to AM as CB to CD.

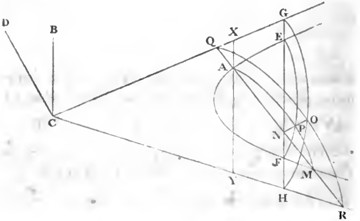
If ABD represent an *oblate* spheroid, the circular section EPF must be made perpendicular to BC; and by the same demonstration APM is an ellipse, whose axis *minor* is AM.

Cor. 1. All sections whose planes are parallel to each other, are similar ellipses, because CB and CD remain unaltered.

Cor. 2. If ABD be an *oblate* spheroid, and KL touch the generating ellipse at the extremity K of the axis major, all sections made by planes passing through KL are similar to the generating ellipse. For KL is parallel to CB and therefore the plane of every such section is parallel to the generating plane in some one position of it.

Prop. IV.

If an hyperbola and it's asymptote revolve about the axis major, the sections of the hyperboloid and cone so generated, made by any plane, are similar figures.



Through A draw XY parallel to the axis minor CB.

Then $QN : GN :: QA : XA$

And $NR : NH :: AR : AY$.

$\therefore QN \times NR : GN \times NH :: QA \times AR : XA \times AY$

Or $QN \times NR : NO^2 :: CD^2 : CB^2$ (Hyp. P. 11. & 12.)

$:: AN \times NM : NP^2$ (P. 18. Cor.)

i. e. the rectangle by the abscissæ has to the square of the semi-ordinate the same ratio in both cases. But that ratio in the section of the cone is a constant ratio, and it is therefore a constant ratio in the section of the hyperboloid. Also the conclusion is the same whether AM cut the hyperbola AFM only, as in the figure, or the two opposite hyperbolas. Hence,

I. If QAR cuts both asymptotes, the section APM is an ellipse.

II. If QR is parallel to one asymptote, APM is a parabola.

III. If QR cuts one asymptote and also the other produced backwards, APM is an hyperbola; and if QR be parallel to the axis major, APM is similar to the generating hyperbola.

Cor. If the cone be cut by a plane which touches the hyperboloid at any point, the section is an ellipse, whose axis minor is always equal to the axis minor of the generating hyperbola and whose axis major$=2CD$.